On the stability of a plane vortex sheet with respect to three-dimensional disturbances

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The previously established criterion for the stability of a vortex sheet with respect to two-dimensional disturbances in a compressible fluid is extended to three-dimensional disturbances, and it is shown that disturbances travelling at sufficiently oblique angles with respect to the undisturbed flow are unstable. Slip-line instability downstream of a Mach reflexion is discussed briefly.

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1. The two-dimensional problem

It has been established (Miles 1958) that a vortex sheet separating two perfect fluids in uniform motion is stable with respect to small, two-dimensional disturbances for sufficiently high values of the relative velocity $|U_+ - U_-|$, where $U = U_\pm$ for $y \gtrless 0$. If the densities and sonic velocities, $\rho_\pm$ and $a_\pm$, are related according to

$$\rho_+ a_+^2 = \rho_- a_-^2, \tag{1}$$

a necessary and sufficient condition for such stability is

$$|U_+ - U_-| > (a_+^{\frac{2}{3}} + a_-^{\frac{2}{3}})^{\frac{3}{2}}. \tag{2}$$

This conclusion loses much of its significance if three-dimensional disturbances are admitted, for then a simple extension of Squire's theorem (Lin 1955) implies that disturbances travelling at sufficiently oblique angles with respect to the uniform flows are unstable.

2. Three-dimensional analysis

Let us assume that the uniform flows are directed along the x-axis and that the vortex sheet exhibits the displacement

$$n(x, z, t) = A \exp [i\alpha(x \cos \phi + z \sin \phi - ct)], \tag{3}$$

relative to its equilibrium position $y = 0$. Small disturbances in the pressure then must satisfy the wave equation

$$a^2 \nabla^2 p = \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)^2 p \tag{4}$$

in $y \gtrless 0$, with appropriate subscripts on a and U, and U enters the boundary conditions only in the calculation of the acceleration

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)^2 n. \tag{5}$$

Requiring p to exhibit the same (x, z, t)-dependence as n in consequence of the assumed linearity, we may reduce (4) and (5) to

$$a^2\left(\frac{\partial^2}{\partial y^2}-\alpha^2\right)p = -\alpha^2(U\cos\phi - c)^2 p \tag{6}$$

and

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = -\alpha^2(U\cos\phi - c)^2 n. \tag{7}$$

We infer from (6) and (7) that the analysis for a two-dimensional disturbance can be generalized to that for a three-dimensional disturbance of the form (3) simply by replacing $U_\pm$ by $U_\pm \cos\phi$. It follows that the stability criterion (2) then must be replaced by

$$|(U_+ - U_-)\cos\phi| > (a_+^{\frac{2}{3}} + a_-^{\frac{2}{3}})^{\frac{3}{2}} \tag{8}$$

for obliquely moving disturbances, and hence that such disturbances will be unstable for

$$|\cos\phi| < |U_+ - U_-|^{-1}(a_+^{\frac{2}{3}} + a_-^{\frac{2}{3}})^{\frac{3}{2}}. \tag{9}$$

3. Slip-line instability

Vortex sheets in compressible flows are observed principally as slip lines emanating from the triple intersections of shock waves (Mach or lambda reflexions) and appear to be stable in the majority of published photographs. Whether such slip lines are ever truly stable, and therefore subject only to laminar diffusion under the action of viscosity and heat conduction, or only weakly unstable appears to be regarded as an open question, although Landau & Lifschitz (1959, p. 407) state tersely that 'as usual, the tangential discontinuity becomes a turbulent region'. Recently, however, Duff (1962) has published photographs of Mach reflexions in both air and CCl_4 vapour ($\gamma \doteqdot 1{\cdot}08$) in which Kelvin–Helmholtz instability of the slip lines definitely can be observed.†

It appears likely, both from the present results and from Duff's observations, that slip-line instability should be the rule, rather than the exception, and that the apparent stability in other observations may be simply a consequence of the very limited downstream extent of the field of view. Further theoretical progress would require the consideration of three-dimensional disturbances of the laminar mixing region separating two real gases in nearly parallel flow (cf. Lin 1955, pp. 100–2).

† Bryson & Gross (1961, figures 3–5) have observed slip lines 'rolling up' into vortices following shock-wave diffraction by a cylinder, but they attribute these vortices to shock-wave boundary-layer interaction.

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